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TRANSITORY SERVICE SYSTEMS

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ABSTRACT

Many (perhaps most) service systems, such as repair and job shops, computation centers, and transportation networks, experience demand that is non-stationary in time. This paper describes models for situations in which demands made are by a finite number of individuals, who, having been served, do not return until much later. Such a transitory demand or arrival process describes many phenomena, among them being commuter rush hours, and also perhaps the effect on a population of individuals their simultaneous exposure to a dosage of medicine, a disease, or even a pollutant. Our paper formulates several models for the service of such demands and describes the manner in which system state may be approximated by Gaussian processes, in particular the Ornstein-Uhlenbeck and Wiener diffusions.

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TRANSITORY SERVICE SYSTEMS

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I. Introduction.

The theory of service systems and waiting lines has developed extensively over a number of years, stimulated by a desire to describe congestion and traffic in many applied situations. Almost without exception, however, the mathematical theory has dealt with stationary models, in which a steady stochastic flow of arrivals approaches a set of servicing facilities; for recent exceptions see the work of Newell [10], and that of Milch [9]. Nevertheless, many quite reasonable situations suggest the study of models that are non-stationary. In this paper we describe and analyze a class of models of a distinctly non-stationary nature. We call them transitory service systems.

Transitory phenomena occur when members of a finite population of customers makes application for service once, or at most a finite number of times. Some examples follow.

(a) A questionnaire, letter, or newspaper or television advertisement offering a prize or job, is transmitted. A certain number of individuals choose to respond, and do so after various times. The responses must then be processed. Of interest is the number of individuals in various stages: e.g. potential responders waiting to respond, and responders undergoing processing. One may also wish to infer the number of potential responders in the population on the basis of the responses during a short initial time period.

(b) A message is sent by radio to a group of stations or ships. The message requires a reply or verification, an act that takes place after a random time, and requires a random time to complete. The probability distribution of the number of messages being verified at a particular moment after transmission may be useful, as are other properties of the reply process.

(c) Certain copies of a new model of automobile or other consumer (or industrial or military) product contain a defect which is possibly dangerous and should be modified. When the defect becomes evident a recall or correction notice is sent out, following which the copies straggle in for modification. We are interested in the number of copies that have applied for, and are undergoing, modification, at any time. We are also interested in the length of time to complete all modifications.

(d) A ship puts out to sea with all systems in operative condition. Realistically, however, each system has a chance of failing during a voyage. Let us assume, as may hopefully often be reasonable, that at most one failure per system will occur during a voyage; it is this assumption that makes our process transitory. When the failure occurs a repair process is initiated. We are interested in total ship unavailability (the number of systems on repair) as a function of time, and other related measures of total system degradation.

(e) It has been suggested [2] that a road or freeway network may be considered to be analogous to an infinite server queueing system, with drivers' times on the freeway identified to be service times. Clearly, there will be certain intervals (rush-hour periods) during which a large segment of the population of automobiles will enter the freeway. They will not return until the next rush hour. Consequently, freeway occupancy appears to be a succession of transitory problems, one for each rush-hour period.

Other examples will occur to the reader. We shall now embark on an analysis of the stochastic models suggested. Our particular emphasis will be upon approximations of a simple type.

II. "Sufficient Server" Models.

Let there be N potential applicants for service in a population. The time of application of customer i, measured from some initial instant, is denoted by T_i. The service time of customer i is S_i. We assume that at application each customer may immediately begin service, i.e. there are "sufficient servers." This is analogous to the infinite server model of conventional theory.

Clearly applicant i is, at time t,

a) not yet in service if $T_i > t$,

b) in service if $T_i < t$, but $T_i + S_i > t$,

c) finished with service if $T_i + S_i < t$.

Consider the following models.

Model 1. $\{T_i\}$ and $\{S_i\}$ independently and identically distributed (i.i.d.).

Let T_i have the distribution F, and S_i the distribution G; let $\bar{F}(t) = 1 - F(t)$, and $\bar{G}(t) = 1 - G(t)$. Then if $A(t)$ denotes the number of applicants who have not yet applied, $Q(t)$ denotes those who have applied and are in service, and $C(t)$ denotes the number of those who are finished at t we can write down directly, on the basis of independence, the joint generating function

$$E[u^{A(t)}\, v^{Q(t)}\, w^{C(t)}] = \{u\bar{F}(t) + v\bar{G}*F(t) + wG*F(t)\}^N. \tag{2.1}$$

where

$$\bar{G}*F(t) = \int_0^t [1 - G(t-x)]dF(x) \tag{2.2}$$

and

$$G*F(t) = \int_0^t G(t-x)dF(x). \tag{2.3}$$

If we put $u = w = 1$ we see that $Q(t)$, the number in service has the generating function

$$E[v^{Q(t)}] = \{[1 - \bar{G}*F(t)] + v\bar{G}*F(t)\}^N \tag{2.4}$$

of the binomial distribution. Therefore the expectation and variance of the number in service at time t are

$$\begin{aligned} E[Q(t)] &= N\int_0^t [1 - G(t-x)]dF(x) \\ Var[Q(t)] &= N\int_0^t [1 - G(t-x)dF(x)\{1 - \int_0^t [1 - G(t-x)]dF(x)\}. \end{aligned} \tag{2.5}$$

As N becomes large one can approximate the distribution of $Q(t)$ by the normal with mean and variance matching (2.5).

The conventional infinite server problem, to which this model compares, yields no such simple time-dependent result. Recall that if arrivals are stationary Poisson then the stationary distribution is also Poisson; see Riordan [12]. Of course, if population size, N, is large then $Q(t)$ <u>may</u> turn out to be approximately Poisson at a fixed time t. Consider the following extensions.

<u>Model 2</u>. $\{T_i\}$ and $\{S_i\}$ i.i.d.; N is Poisson.

If population size N is a random variable with the Poisson distribution of mean a, then clearly $Q(t)$ is also Poisson with mean $a\bar{G}*F(t)$, as follows from the generating function:

$$E[v^{Q(t)}] = \sum_{n=0}^{\infty} \{1 - \bar{G}*F(t) + v\bar{G}*F(t)\}^n e^{-a} \frac{a^n}{n!}$$
$$= \exp\{a\bar{G}*F(t)(1-v)\}. \tag{2.6}$$

Moreover, $A(t)$, $Q(t)$, and $C(t)$ are jointly and independently Poisson in this model as is evident from the fact that the generating function factors:

$$E[u^{A(t)}v^{Q(t)}w^{C(t)}] = \exp\{a\bar{F}(t)(1-u)\}\exp\{a\bar{G}*F(t)(1-v)\}$$
$$\exp\{aG*F(t)(1-w)\}. \tag{2.7}$$

It cannot, however, be concluded that $\{Q(t), t \geq 0\}$ is a Poisson <u>process</u>.

<u>Model 3</u>. $\{T_i\}$ are i.i.d. Gamma; $\{S_i\}$ are i.i.d.; the population size N becomes large.

Suppose the arrival distribution, that of T_i, is Gamma, with density

$$f(t) = e^{-\lambda t} \frac{(\lambda t)^{k-1}\lambda}{\Gamma(k)} \tag{2.8}$$

for $t \geq 0$ and $k > 1$. Then we can express the generating function of $Q(t)$, (2.4), as

$$E[v^{Q(t)}] = [1 - (1-v) \int_0^t [1 - G(t-x)]e^{-\lambda x} \frac{N(\lambda x)^{k-1}}{N\Gamma(k)} dx]^N \qquad (2.9)$$

Now scale time by letting $\lambda^k N = \theta$, a constant, and allow $N \to \infty$. Then

$$E[v^{Q(t)}] \to \exp\{-(1-v)\theta \int_0^t [1 - G(t-x)] \frac{x^{k-1} dx}{\Gamma(k)} \} \qquad (2.10)$$

The continuity theorem then implies that under the stated conditions the distribution of $Q(t)$ approaches the Poisson with paramters $\theta \int_0^t [1 - G(t-x)] \frac{x^{k-1} dx}{\Gamma(k)}$. Notice that if $k = 1$ then in the limit we have the familiar "infinite server" solution of the classical theory: if arrivals are a stationary Poisson process--to which the process $\{A(t), t\geq 0\}$ converges as $N \to \infty$--then $\{Q(t), t\geq 0\}$ is a non-stationary Poisson process with $E[Q(t)] = \theta \int_0^t [1-G(x)]dx \to \theta E[S]$ as $t \to \infty$. For other k values the process behaves in a comparable fashion. It may be contended that the type of large-N result just derived actually justifies the use of the classical queueing models that assume Poisson arrivals with stationary independent increments, at least early in the arrival process. Of course, an approximation based upon $N\lambda = \theta$ cannot be accurate late in the process, e.g. after a time $t_{1/2} = \frac{\ell n\ 2}{\lambda}$, when on the average one-half of the calling population has arrived. Therefore, in the following section we develop a procedure for approximating our processes $A(t)$, $Q(t)$, and $C(t)$ by Ornstein-Uhlenbeck processes; these approximations can be expected to be reasonably accurate for moderate N, and t-values of practical interest.

III. Diffusion Approximation for Sufficient Servers.

Since $A(t)$, the number of arrivals in time t, and $Q(t)$, the number in service, are approximately normally distributed for large N, we are led to consider the use of a diffusion approximation for these processes; see Feller [4], and Cox and Miller [3]. We remind the reader that a diffusion is a Gaussian process with continuous sample paths. We next proceed to develop approximating diffusions for certain transitory systems, and then to make use of such approximations to obtain properties of the service systems. Our approach here is in the spirit of McNeil [8]; other authors have utilized similar procedures. For example, see Anderson and Darling [1], and Whittle [14].

III-1. Approximating the Arrival Process.

Let $A_N(t)$ denote the number of arrivals by t out of a population of N. Each arrival time is independently distributed with d.f. $F(t)$, density function $f(t)$, and hazard rate $\lambda(t) = f(t)[1-F(t)]^{-1}$.

Now clearly $\frac{A_N(t)}{N} \to F(t)$ in probability as $N \to \infty$ for every fixed t. In fact, the convergence is uniform with probability one by the Glivenko-Cantelli theorem; see Tucker [13], p. 127. Hence, we are led first to approximate $A_N(t)$ by a deterministic component, $x(t)$, for large N:

$$A_N(t) \approx Nx(t) \tag{3.1}$$

Noting that if $A_N(t)$ arrivals have occurred by t, and thus that the probability of another arrival in $(t,t+dt)$ is essentially $[N - A_N(t)]\lambda(t)dt$ we can derive

$$\frac{dx}{dt} = \lambda(t)[1 - x(t)], \tag{3.2}$$

the solution of which is $x(t) = F(t)$.

To include a stochastic element, or noise, study

$$S_N(t) = \frac{A_N(t) - Nx(t)}{\sqrt{N}} . \tag{3.3}$$

Our first approach is to derive the transformed version of the forward differential equation satisfied by the density function of S_N, pass to the limit as $N \to \infty$, and thus show that if S_N converges to a limiting process the latter must be non-stationary Ornstein-Uhlenbeck; see Cox and Miller [3].

If

$$\phi_N(\theta,t) = E[e^{i\theta A_N(t)}] \qquad i = \sqrt{-1}, \tag{3.4}$$

is the characteristic function (ch.f.) of $A_N(t)$ we have

$$E[e^{i\theta A_N(t+dt)}] = E[e^{i\theta A_N(t)}\{1 - [N-A_N(t)]\lambda(t)dt\}]$$

$$+ E[e^{i\theta(A_N(t)+1)}\{N-A_N(t)\}\lambda(t)dt] + o(dt), \tag{3.5}$$

and upon collecting terms and letting $dt \to 0$ there emerges the differential equation

$$\frac{\partial \phi_N(t)}{\partial t} = (e^{i\theta}-1)\{N\phi_N(\theta,t) + i\frac{\partial \phi_N}{\partial \theta}\}\lambda(t) \tag{3.6}$$

Now to derive an equation for the ch.f. of $S_N(t)$,

$$\psi_N(t) = E[e^{i\theta S_N(t)}], \tag{3.7}$$

we note that

(i) $\psi_N(\theta,t) = e^{-i\theta x(t)\sqrt{N}}\, \phi_N(\theta/\sqrt{N},t)$,

(ii) $\frac{\partial \phi_N}{\partial t}(\theta/\sqrt{N},t) = e^{i\theta x(t)\sqrt{N}}\frac{\partial \psi_N(\theta,t)}{\partial t} + i\theta x'(t)\sqrt{N}\, e^{i\theta x(t)\sqrt{N}}\, \psi_N(\theta,t)$,

$$\text{(iii)}\quad \frac{\partial\phi_N}{\partial\theta}(\theta/\sqrt{N},t) = e^{i\theta x(t)\sqrt{N}}\,\frac{\partial\psi_N}{\partial\theta}(\theta,t) + ix(t)\sqrt{N}\,e^{i\theta x(t)\sqrt{N}}\,\psi_N(\theta,t). \tag{3.8}$$

Next change θ to $\theta' = \theta/\sqrt{N}$ in (3.5), utilize (3.7), expand the exponentials $(e^{i\theta/\sqrt{N}} - 1 = \frac{i\theta}{\sqrt{N}} - \frac{\theta^2}{2N} + O(\frac{1}{N^{3/2}}))$ and collect terms. The essential outcome is

$$\frac{\partial\psi_N}{\partial t} + \psi_N i\theta\sqrt{N}\{x'(t) - \lambda(t)[1-x(t)]\} = -\frac{\theta^2}{2}\,\lambda(t)[1-x(t)]\psi_N$$
$$- \theta\lambda(t)\,\frac{\partial\psi_N}{\partial\theta} + o(\frac{1}{\sqrt{N}}). \tag{3.9}$$

In order for the equation to be satisfied by a limiting ch.f., $\psi(\theta,t)$, as $N \to \infty$ the term in brackets must vanish; solution of the resulting differential equation once again presents the deterministic equation of (3.2). There remains the equation

$$\frac{\partial\psi}{\partial t} = -\theta\lambda(t)\,\frac{\partial\psi}{\partial\theta} - \frac{\theta^2}{2}\,f(t)\psi \tag{3.10}$$

which is the transformed version of the forward equation for a non-stationary Ornstein-Uhlenbeck (O.U.) process; see Cox and Miller [3], pp. 218-219. The infinitesimal mean and variance are respectively $\beta(x,t) = -x\lambda(t)$ and $\alpha(x,t) = f(t)$. Since the O.U. process is Gaussian we may differentiate the Gaussian ch.f. $\exp[i\theta\mu(t) - \frac{\theta^2}{2}\,\sigma^2(t)]$ and equate coefficients in (3.9) to obtain the mean and variance. Not surprisingly,

$$E[S(t)] = 0, \qquad \text{and} \quad \mathrm{Var}[S(t)] = F(t)[1-F(t)] \tag{3.11}$$

Hence we are led to approximate $A_N(t)$ by

$$A_N(t) \approx NF(t) + \sqrt{N}\,S(t) \tag{3.12}$$

where $S(t)$ is an O.U. process with parameters given by (3.11).

An alternative derivation of the O.U. parameters is easy, once one settles upon a diffusion approximation. The stochastic differential equation for A_N is given approximately by

$$dA_N(t) = \lambda(t)[N-A_N(t)]dt + Z(t)\sqrt{\lambda(t)[N-A_N(t)]dt}, \tag{3.13}$$

$Z(t)$ being a "purely random process," or the "derivative" of a Wiener process. The first right-hand term of (3.12) is the deterministic component or drift, and the second is the supplementary randomness or noise, the scale of which follows from considering possible events in a dt interval. Now apply normalization (3.3) to obtain

$$dS_N + \sqrt{N}[x'(t) - \lambda(t)\{1-x(t)\}] = -\lambda(t)S_N(t)dt$$

$$+ Z(t)\sqrt{\lambda(t)\left[1-x(t) - \frac{S_N}{\sqrt{N}}\right]dt}, \tag{3.14}$$

and once again we are compelled to set $x'(t) = \lambda(t)[1-x(t)]$ and to realize that the limiting noise satisfies

$$dS(t) = -\lambda(t)S(t) + Z(t)\sqrt{f(t)dt}, \tag{3.15}$$

the stochastic differential equation for the O.U. process with drift and variance parameters already derived.

III-2. Passage Probabilities for the Arrival Process.

In this section we show how the limiting O.U. process furnishes an approximation to the probability that $A_N(t) \le \mathcal{B}(t)$ for <u>all</u> t, $\mathcal{B}(t)$ being a suitably selected boundary. Our asymptotic result is closely related to the one-sided Kolmogorov-Smirnov test, and may be used for the same purpose, i.e. as a test of the statistical hypothesis that the arrival distribution $F(t)$ generates arrivals more rapidly than does $F_0(t)$, i.e. that $F(t) \geq F_0(t)$ for all t. The methodology employed may also be used to estimate the maximum queue size during a single rush period, but further approximations are required and this problem will be attacked later.

It is shown in [3] that an O.U. process, $S(t)$ may be expressed in terms of a Wiener process, $X(t)$:

$$S(t) = a(t)X\{\tau(t)\} \tag{3.16}$$

where $\tau(t)$ represents a time scale transformation, and $a(t)$ is a real differentiable function. In order to represent our particular process we have

$$\frac{a'(t)}{a(t)} = -\lambda(t), \quad \text{and} \quad a^2(t)\tau'(t) = f(t). \tag{3.17}$$

Solving, we find that

$$a(t) = 1 - F(t), \quad \text{and} \quad \tau(t) = \frac{F(t)}{1 - F(t)}\,. \tag{3.18}$$

Hence

$$\frac{S(t)}{1 - F(t)} = X\{\frac{F(t)}{1 - F(t)}\} \equiv X(\tau) \tag{3.19}$$

$X(\tau)$ being a Wiener process with zero drift and infinitesimal variance τ. It follows that if

$$X(\tau) \leq B(\tau), \quad \text{for} \quad \{\tau : 0 \leq \tau \leq F(T)[1-F(T)]^{-1}\}$$

then

$$\frac{S(t)}{1 - F(t)} \leq B(\tau(t)) \quad \text{for} \quad \{t: 0 \leq t \leq T\}.$$

We consider certain special cases results for which are at hand.

Linear Boundaries. Let $B(\tau) = \alpha + \beta\tau$, $\forall\tau \geq 0$; $\alpha,\beta > 0$. Then it is known that

$$P\{X(\tau) \leq \alpha + \beta\tau,\ \forall\tau \geq 0\} = 1 - e^{-2\alpha\beta} \tag{3.20}$$

Hence

$$S(t) \leq \alpha[1-F(t)] + \beta F(t), \quad \forall t \geq 0$$

with probability $1 - e^{-2\alpha\beta}$, and furthermore

$$A_N(t) \leq NF(t) + \sqrt{N}\{\alpha[1-F(t)] + \beta F(t)\}, \quad \forall t \geq 0 \tag{3.21}$$

with approximate probability $1 - e^{-2\alpha\beta}$.

Linear Boundaries, Finite Stopping. Let $B(\tau)$ be as above; then if X is a Wiener process and $\tau > 0$ is fixed,

$$P\{X(\tau') < \alpha + \beta\tau',\ \forall\ \tau' < \tau,\ X(\tau) \leq x < a\} =$$

$$\Phi\left(\frac{x-\beta\tau}{\sqrt{\tau}}\right) - e^{2\alpha\beta}\Phi\left(\frac{x-\beta\tau-2\alpha}{\sqrt{\tau}}\right) \tag{3.22}$$

where Φ is the standard normal distribution; see Cox and Miller [3], pp. 220-221. Now use (3.19):

$$P\{S(t) \leq \alpha[1-F(t)] + \beta F(t),\ \forall\ t \leq T,\ S(T) \leq y\} =$$

$$\Phi\left[\frac{y(1+\tau)-\beta\tau}{\sqrt{\tau}}\right] - e^{-2\alpha\beta}\ \Phi\left[\frac{y(1+\tau)-\beta\tau-2\alpha}{\sqrt{\tau}}\right] \tag{3.23}$$

where $\tau = \dfrac{F(T)}{1-F(T)}$, $T = F^{-1}\left(\dfrac{\tau}{1+\tau}\right)$.

Thus we can represent the probability that the noise component at time T is in any Borel set of $(-\infty, \ \alpha[1-F(T)] + \beta F(T))$ for any finite T. This result can be carried over to an approximation to the distribution of $A_N(t)$, subject to boundary avoidance prior to T.

Similar problems involving two boundaries are also solvable. In addition, the distribution of the time required to cross a boundary of the form $\alpha[1-F(t)] + \beta F(t)$ can be derived, although moments are not usually expressible in a simple closed form.

Application of the above material to a problem in sequential testing is described in the thesis by Gwinn [5], and an expanded treatment is in preparation.

III-2. Diffusion Approximation of $(A_N(t), Q_N(t))$.

In order to establish a diffusion approximation for $Q_N(t)$, the number undergoing service, we must treat the bivariate process $(A_N(t), Q_N(t))$, focusing on the stochastic elements

$$S_N(t) = \frac{A_N(t) - Nx(t)}{\sqrt{N}}$$

and

$$T_N(t) = \frac{Q_N(t) - Ny(t)}{\sqrt{N}} \tag{3.24}$$

We first proceed as was done earlier. Letting $\lambda(t) = f(t)[1-F(t)]^{-1}$ denote an individual's arrival rate, and assuming exponential service times, we write

$$P\{A_N(t) = n, Q_N(t) = j \,|\, A_N(t) = n, Q_N(t) = j\} = 1-\{\lambda(t)[N-n]+\mu j\}dt+o(dt)$$

$$P\{A_N(t) = n+1, Q_N(t) = j+1 | A_N(t) = n, Q_N(t) = j\} = \lambda(t)[N-n]dt+o(dt)$$

$$P\{A_N(t) = n, Q_N(t) = j-1 | A_N(t) = n, Q_N(t) = j\} = \mu j dt+o(dt). \qquad (3.25)$$

other probabilities being negligible.

Next form the characteristic function expression

$$E[e^{i\theta_1 A_N(t+dt)+i\theta_2 Q_N(t+dt)}] = E[e^{i\theta_1 A_N(t)+i\theta_2 Q_N(t)}\{1-\lambda(t)[N-A_N(t)]\}x$$

$$\{1-e^{i\theta_1+i\theta_2}\} - \mu Q_N(t)\{1-e^{-i\theta_2}\} + o(dt) \qquad (3.26)$$

Then substitute from (3.24), divide by $dt \to 0$, and substitute as in (3.8), letting $N \to \infty$. In order for a limiting ch.f. ψ to exist for (S_N, T_N) it turns out that

$$x'(t) = \lambda(t)[1-\lambda(t)] = f(t)$$

and (3.27)

$$y'(t) = \lambda(t)[1-x(t)] - \mu y(t)$$

and solution gives the deterministic components of the previous section. Finally, ψ satisfies

$$\frac{\partial\psi}{\partial t} = -\frac{1}{2}[(\theta_1+\theta_2)^2 f(t) + \theta_2^2\mu y(t)]\psi - (\theta_1+\theta_2)\lambda(t)\frac{\partial}{\partial\theta_1}\psi - \theta_2\mu\frac{\partial}{\partial\theta_2}\psi \qquad (3.28)$$

This partial differential equation is the Fourier transformed version of a bivariate non-stationary Ornstein-Uhlenbeck process.

A stochastic differential equation approach runs as follows. Approximately,

$$dQ_N(t) = \{\lambda(t)[N-A_N(t)] - \mu Q_N(t)\}dt + Z(t)\sqrt{\mu(t)[N-A_N(t)]dt+ Q_N(t)dt}, \qquad (3.29)$$

so when (3.24) is applied and $N \to \infty$ we find that the deterministic part, y, must satisfy

$$y' = f(t) - \mu y$$

or (3.30)

$$y(t) = \int_0^t e^{-\mu(t-s)} f(s)ds$$

in accordance with earlier results, while the limiting noise is described by

$$dT = -[\lambda(t)S(t) + \mu T(t)]dt + Z(t)\sqrt{[f(t)+\mu y(t)]dt} \tag{3.31}$$

Thus T is conditionally Ornstein-Uhlenbeck, given $S(t)$. Also, the purely random parts of dT and dS are correlated, as is clear from the fact that an additional arrival lengthens the queue; the correlation term is $f(t)$, and we can write

$$dS(t) = -\lambda(t)S(t)dt + Z_1(t)\sqrt{f(t)dt}$$

$$dT(t) = -[\mu T(t)+\lambda(t)S(t)]dt + Z_1(t)\sqrt{f(t)dt} + Z_2(t)\sqrt{\mu y(t)dt} \tag{3.32}$$

where Z_1 and Z_2 represent independent purely random components, i.e. Wiener process derivatives.

III-3. Moments.

Since $\{S(t),T(t)\}$ is a bivariate diffusion with zero mean we know that ψ is of the Gaussian form

$$\psi(\theta_1,\theta_2,t) = \exp\{-\tfrac{1}{2}\,\underline{\theta}'\,\underline{\Sigma}\,\underline{\theta}\} \tag{3.33}$$

where $\underline{\theta}' = (\theta_1,\theta_2)$, and $\underline{\Sigma}$ is the covariance matrix. Hence, we need merely equate coefficients of $\theta_i\theta_j$ $(i,j = 1,2)$ in order to derive the covariance matrix elements. Not surprisingly, these agree with earlier exact results:

$$\mathrm{Var}[S(t)] = x(t)[1-x(t)]; \quad \mathrm{Var}[T(t)] = y(t)[1-y(t)]$$

$$\equiv F(t)[1-F(t)]$$

$$\mathrm{Cov}[S(t),T(t)] = [1-F(t)] \int_0^t e^{-\mu(t-s)} f(s)ds. \tag{3.34}$$

IV. Single-Server Models.

Many of the transitory situations that are encountered in practice actually involve competition for facilities; e.g. for repairmen, clerks, freeway space, or computers. Therefore in this section we describe models for a single-server system confronted by transitory demand.

Once again, a bivariate stochastic process is required to describe our model. If N individuals wait to make application for service, then the number of new arrivals in $(t,t+dt)$ depends upon the random number that has already appeared by time t. The waiting line development is, of course, influenced by the service rate as well. It is the coordinated effects of these processes that we represent, and attempt to make comprehensible by means of approximations.

Model 1. Markovian Arrivals and Exponential Service.

Let $A(t)$ be the number of arrivals that have occurred by t, and $Q(t)$ be the number waiting and in service at t at the service facility. Then let

$$P_{jn}(t) = P\{Q(t) = j, A(t) = n \mid A(0) = 0, Q(0) = 0\}, \qquad (4.1)$$

for $0 \leq j \leq n \leq N$. Given $A(t) = n$, let $\lambda_n dt$ be the probability of exactly one arrival in $(t,t+dt)$, and let μdt be the probability of exactly one departure in the same interval. The usual independence assumptions, cf. Feller [4], are made. Then we can write down the forward differential equations for $\{P_{jn}(t)\}$:

$$P'_{00}(t) = -\lambda_0 P_{00}(t)$$

$$P'_{jn}(t) = -(\lambda_n+\mu)P_{jn}(t) + \mu P_{j+1,n}(t) + \lambda_{n-1} P_{j-1,n-1}(t),$$

$$j = 1,2,\ldots,n-1;$$

$$P'_{0n}(t) = -\lambda_n P_{0n}(t) + \mu P_{1n}(t)$$

$$P'_{nn}(t) = -(\lambda_n+\mu)P_{nn}(t) + \lambda_{n-1} P_{n-1,n-1}(t), \tag{4.2}$$

where initial conditions, e.g. $P_{00}(0) = 1$, and $P_{jn}(0) = 0$ otherwise must be specified. The arguments leading to (4.2) are standard and will be omitted. Of course, one can also allow arrival and departure rates to depend upon t.

For any explicitly specified λ_n the solution to (4.2) can be computed numerically, for (4.2) is only a system of finitely many differential equations. If desired, one can Laplace transform throughout, but there is no simple explicit formula even for the transform. See the paper by Perlas, [11], for further details.

Model 1-A. Independent Exponential Arrivals.

A model of some interest assumes that $\lambda_n = \lambda(N-n)$, or, equivalently, that each individual's arrival time is independent with distribution $1 - e^{-\lambda t}$. Then a deterministic approximation, that one would be tempted to use for N large, is suggested:

$$d\underline{x}(t) = \lambda[N-\underline{x}(t)]dt \tag{4.3}$$

and

$$\underline{y}(t+dt) = \max\{\underline{y}(t) + \lambda[N-\underline{x}(t)]dt - \mu dt, 0\}, \tag{4.4}$$

where $\underline{x}$ and $\underline{y}$ represent deterministic approximations to A and Q. Solutions, subject to $A(0) = Q(0) = 0$, are

$$\underline{x}(t) = N[1-e^{-\lambda t}]$$

and

$$\underline{y}(t) = \max\{N(1-e^{-\lambda t}) - \mu t, 0\}. \tag{4.4}$$

More will be said about these approximations in the next section.

Numerical solutions of the above equations have been carried out to compare the total waiting times (i) under the full stochastic model, (4.1), and (ii) under the deterministic approximation (4.2), (4.3). These are summarized in Tables 1 and 2, which show that the deterministic approximation improves greatly as $N \to \infty$. Not surprisingly, the deterministic approximation falls below the stochastic expectation.

IV-1. The Probability of No Wait.

In [6], Kabak describes a transitory problem in which attention focuses upon the probability that no arrival must wait. Other applications for this model may well exist, e.g. in military situations. Let us suppose that calls occur in accordance with the pure birth process of Model 1, and let successive service times $\{S_n, \; n = 1,2,\ldots N\}$ be independent but arbitrarily distributed in accordance with $F_n(x)$. If $T_{(i)}$ represents the time until the $i^{\underline{th}}$ arrival occurs, then the probability of no wait is seen to be that of the following event

$$T_{(2)}>S_1, T_{(3)}-T_{(2)}>S_2, T_{(4)}-T_{(3)}>S_3, \ldots, T_{(N)}-T_{(N-1)}>S_{N-1} \tag{4.5}$$

Since times between successive arrivals are independently exponential in this Markovian model it is immediate that the probability of no wait is

$$P = \prod_{j=1}^{N-1} \int_0^\infty e^{-\lambda_j t}\, dF_j(t) = \prod_{j=1}^{N-1} \hat{F}_j(\lambda_j) \tag{4.6}$$

where $\hat{F}$ is the Laplace-Stieltjes transform of F.

In the case of Model 1-A this specializes to

$$P = \frac{\mu}{\mu+\lambda_1} \cdot \frac{\mu}{\mu+\lambda_2} \cdots \frac{\mu}{\mu+\lambda_{N-1}}$$

$$= \frac{\left(\frac{\mu}{\lambda}\right)^{N-1}}{\left(\frac{\mu}{\lambda}+N-1\right)\left(\frac{\mu}{\lambda}+N-2\right)\ \cdots\ \left(\frac{\mu}{\lambda}+1\right)} \tag{4.7}$$

$$\sim \frac{\left(\frac{\mu}{\lambda}\right)^{N} \Gamma\left(\frac{\mu}{\lambda}\right)}{(N-1)!\ (N-1)^{\frac{\mu}{\lambda}}} \qquad \text{as} \quad N \to \infty \tag{4.7}$$

according to Euler's product definition of the gamma function; see Knopp [7]. Of course, if service times are constant rather than exponential a simple exact solution is possible:

$$P = e^{-\frac{1}{\mu}\sum_{j=1}^{N-1} \lambda(N-j)} = e^{-\frac{\lambda}{\mu}\frac{N(N-1)}{2}}$$

In the event that F is a one-sided stable distribution of order α, $0 < \alpha < 1$, we have essentially

$$P = e^{-\left(\frac{\lambda}{\mu}\right)^{\alpha} \sum_{j=1}^{N-1} (N-j)^{\alpha}} \sim e^{-\left(\frac{\lambda}{\mu}\right)^{\alpha} \frac{N^{1+\alpha}}{1+\alpha}} \tag{4.9}$$

Thus even though the stable laws have right tails long enough to engender infinite first moments, the mass near zero governs the above probability.

IV-2. The Duration of the Process.

If we let τ denote the time to complete the process of arrival, queueing, and service, then it is clear from (4.1) that

$$P\{\tau \leq t\} = P_{0N}(t). \tag{4.10}$$

Unfortunately, this distribution has no neat closed form, although numerical properties can be derived by solution of (4.2).

It is worth noting that for large N and under the assumption of Model I-A (4.10) is very close to a gamma distribution with mean $\frac{N}{\mu}$ and variance $\frac{N}{\mu^2}$. The explanation is that for large N a queue quickly develops behind the server, and does not vanish until the last service occurs. The server effectively continues to be active through the service of all N arrivals.

V. Diffusion Approximation for Single-Server Models.

The setup to be analyzed now is as follows. N individuals independently choose their arrival times from a distribution $F(x)$, the latter possessing a smooth density $f(x)$. In the order of their arrival these are served at a single servicing facility; service times are exponential at rate $N\mu$. Let $A_N(t)$ denote the number of arrivals that have occurred on or before t, $Q_N(t)$ is the number awaiting or undergoing service at t, and $D_N(t) = \int_0^t Q_N(s)ds$ is the total delay accumulated up to time t. We shall analyze the birth and death process $\{A_N(t),Q_N(t)\}$; if F is exponential this is just the process discussed in the last section. A useful approximate approach is to derive a diffusion approximation, valid for large N. In the present approximation, $F(t) > \mu t$ for a substantial range of t-values (heavy traffic) is also a requirement.

V-1. Deterministic Approximation of $\{A_N(t),Q_N(t),D_N(t)\}$.

Begin by isolating the deterministic component of the process $Q_N(t)$, since that of $A_N(t)$ has already been treated. Notice that if $\frac{Q_N(t)}{N} \to y(t)$ as $N \to \infty$ and it is assumed that y has a derivative, then

$$Q_N(t+h) - Q_N(t) = \lambda(t)[N-A_N(t)]dt - N\mu dt + o(dt) \tag{5.1}$$

if $Q_N > 0$, while

$$Q_N(t+h) - Q_N(t) = \lambda(t)[N-A_N(t)]dt + o(dt) \tag{5.2}$$

if $Q_N = 0$. Dividing by dt, and by N, and taking the limit as $dt \to 0$ and $N \to \infty$ yields the differential equation

$$y'(t) = \lambda(t)[1-x(t)] - \mu h[y(t)] \tag{5.3}$$

where $y(0) = 0$ and

$$h(x) = \begin{cases} 1, & x > 0 \\ 0, & x \le 0; \end{cases} \tag{5.4}$$

the solution is

$$y(t) = \max\{F(t) - \mu t, 0\}. \tag{5.5}$$

It also follows that $z(t)$, the deterministic component of $D(t)$, is

$$z(t) = \int_0^t y(s)ds \tag{5.6}$$

V-2. Diffusion Approximations.

Next define the supplementary randomness or noise components of our three processes:

$$S_N(t) = \frac{A_N(t) - Nx(t)}{\sqrt{N}},$$
$$T_N(t) = \frac{Q_N(t) - Ny(t)}{\sqrt{N}} \tag{5.7}$$
$$U_N(t) = \frac{D_N(t) - Nz(t)}{\sqrt{N}}$$

Begin by writing down the infinitesimal transition probabilities:

$$P\{A_N(t+dt) = A_N(t)\ ,\ Q_N(t+dt) = Q_N(t), D_N(t+dt) = D_N(t)+dtQ_N(t) \mid$$
$$A_N(t), Q_N(t), D_N(t)\} = 1-\lambda(t)[N-A_N(t)]dt-\mu Nh[Q_N(t)]dt+o(dt),$$

$$P\{A_N(t+dt) = A_N(t)+1, Q_N(t+dt) = Q_N(t)+1, D_N(t+h) = D_N(t)+dt[Q_N(t)+\tfrac{1}{2} \mid$$

$$A_N(t), Q_N(t), D_N(t)\} = \lambda(t)[N-A_N(t)]dt+o(dt)$$

$$(5.8)$$

$$P\{A_N(t+dt) = A_N(t), Q_N(t+dt) = [Q_N(t)-1]h[Q_N(t)], D_N(t+dt)$$

$$= D_N(t)+dt[Q_N(t)-\tfrac{1}{2}] \mid A_N(t), Q_N(t), D_N(t)\} = \mu N h[Q_N(t)]dt+o(dt)$$

where once again $h(\cdot)$ represents the unit step at the origin. Now put for the joint ch.f.

$$\psi_N(\theta_1,\theta_2,\theta_3;t) = E[e^{i\theta_1 S_N(t)+i\theta_2 T_N(t)+i\theta_3 U_N(t)}] \qquad (5.9)$$

In order to derive a partial differential equation for ψ_N follow the conditioning argument used earlier on $A_N(t)$ in section III. Also, assume throughout that $h[Q_N(t)] = 1$, so our approximate solutions will not apply when Q_N is small (we must avoid the very end of the process, and for some distributions, F, the very beginning). Omitting the tedious algebra we state that if a limiting noise distribution exists its ch.f. $\psi = \lim\limits_{N\to\infty} \psi_N$, must satisfy the differential equation

$$\frac{\partial\psi}{\partial t} = -\frac{1}{2}[f(t)(\theta_1+\theta_2)^2 + \mu\theta_2^2]\psi - \lambda(t)(\theta_1+\theta_2)\frac{\partial}{\partial\theta_1}\psi + \theta_3\frac{\partial}{\partial\theta_2}\psi \qquad (5.10)$$

This equation is the Fourier-transformed forward equation of a trivariate non-stationary diffusion. As we already know, $S(t)$ is non-stationary O.U. Conditionally upon $S(t)$, $T(t)$ is a non-stationary Wiener process. This is independently established by deriving the stochastic differential equation:

$$dQ_N = \{\lambda(t)[N-Nx(t)-\sqrt{N}S_N(t) - N\mu\}dt$$

$$+ Z(t)\sqrt{\{\lambda(t)[N-Nx(t)-\sqrt{N}S_N(t)+N\mu\, dt} \qquad (5.11)$$

$$= \sqrt{N}\, dT_N + Ndy$$

After passage to the limit as $N \to \infty$ and excision of the deterministic component the expression

$$dT = -\lambda(t)S(t)dt + Z'(t)\sqrt{\{f(t)+\mu\}dt} \qquad (5.12)$$

results. We note that (5.10) may be derived directly from (3.15) and (5.12), the stochastic differential equations for the arrival and queue noise components. Forgetting about the U component, write

$$\psi(\theta_1,\theta_2,t+dt) - \psi(\theta_1,\theta_2,t) =$$

$$E[e^{i\theta_1\{S(t)+dS\}+i\theta_2\{T(t)+dt\}} - e^{i\theta_1 S(t)+i\theta_2 T(t)}]$$

$$= E[\{e^{i\theta_1 dS+i\theta_2 dt} - 1\}\{e^{i\theta_1 S(t)+i\theta_2 T(t)}\}].$$

Now make use of (3.15) and (5.12), and the additional fact that the purely random components are correlated:

$$\psi(\theta_1,\theta_2,t+dt)-\psi(\theta_1,\theta_2,t) = E[\{e^{-i\theta_1\lambda(t)S(t)dt-i\theta_2\lambda(t)S(t)dt}$$

$$e^{-\frac{1}{2}\{f(t)(\theta_1+\theta_2)^2+\mu\theta_2\}dt} -1\}\text{x } e^{i\theta_1 S(t)+i\theta_2 T(t)}] \qquad (5.14)$$

Next expand exponents to order dt, divide by dt, and let $dt \to 0$; after a little rearrangement (5.14) turns into (5.10), with $\theta_3 = 0$ and hence the last term removed. Note that the purely random parts of S and T are correlated with covariance equal to $f(t)dt$. Thus one might write the equations as follows:

$$dS = -\lambda(t)S(t)dt + Z_1(t)\sqrt{f(t)dt}$$

$$dT = -\lambda(t)S(t)dt + Z_1(t)\sqrt{f(t)dt} + Z_2(t)\sqrt{\mu dt} \qquad (5.15)$$

where Z_1 and Z_2 are independent and purely random. It now follows-- as it does also by setting $\theta_1 = -\theta_2$, and $\theta_3 = 0$ in (5.10)-- that the noise process $T(t) - S(t)$ is a Wiener process with zero drift and infinitesimal variance μ.

In summary, our assumptions (which are akin to those of heavy traffic theory in the sense that boundary effects are essentially ignored) imply that arrivals, number in system, and total delay are jointly non-stationary Gaussian. The actual covariance function will be derived subsequently, and some further approximations suggested. From (5.15) it is easily seen that a simulation of our approximation can be carried out by discretizing time: if $t = n\Delta$, $n = 1,2,\ldots,$ then

$$S((n+1)\Delta) = S(n\Delta) - \lambda(n\Delta)S(n\Delta)\Delta + Z_1(n)\sqrt{f(n\Delta)\Delta}$$

$$T((n+1)\Delta) = T(n\Delta) - \lambda(n\Delta)S(n\Delta)\Delta + Z_1(n)\sqrt{f(n\Delta)\Delta} + Z_2(n)\sqrt{\mu\Delta} \qquad (5.16)$$

where here $\{Z_1(n)\}$ and $\{Z_2(n)\}$ are mutually independent sequences of independent $N(0,1)$ random variables, and $S(0) = T(0) = 0$. Notice too that the derivation of (5.12) may be carried out if the service rate, μ, is a sufficiently smooth function of t.

V-3. Moments.

The diffusion process characterized by (5.10) is zero mean trivariate Gaussian, so

$$\psi(\theta_1,\theta_2,\theta_3,t) = \exp\{-\frac{1}{2}\,\underline{\theta}'\underline{\Sigma}\,\underline{\theta}\}. \tag{5.17}$$

where $\underline{\theta}' = [\theta_1,\theta_2,\theta_3]$, $\underline{\theta}$ its transpose, and $\underline{\Sigma}$ the covariance matrix. Now if we differentiate through with respect to t and θ_i, in accordance with (5.10) and equate coefficients of $\theta_i\theta_j$, we obtain the following moments

$$\mathrm{Var}[S] = F(t)[1-F(t)], \quad \mathrm{Var}[T] = F(t)[1-F(t)] + \mu t$$

$$\mathrm{Cov}[S,T] = F(t)[1-F(t)]. \tag{5.18}$$

Even more easily, these moments can be obtained from first principles, making use of the fact that $\mathrm{Var}[T-S] = \mu t$ as implied by (5.15). The fact that the boundary at zero has been neglected, and hence that our model is inadequate late in the process, becomes evident in the expression for $\mathrm{Var}[T]$, which should apparently approach zero with large t, but does not. The model should however be accurate near the peak of the process.

VI. Conclusion.

The purpose of this paper is to formulate and explore a new class of non-stationary service models. We have shown that under certain conditions our models may be described by Gaussian diffusion processes, and have hinted at the manner in which known results for such diffusion may be utilized to study model properties; for examples we exhibit boundaries below which the process moves with prescribed probability, and derive equations for the total time spent waiting by all arrivals. Although all technical details of approximation accuracy are not yet well understood, further theoretical and numerical explorations are under way. In any case, the diffusion approximations suggested appear to make an offer that the modeler of stochastic phenomena can't refuse.

A brief mention of further related questions that deserve attention would include (i) the (approximate) distribution of $\max_{0 \le t \le \infty} Q(t)$, both in sufficient and single server contexts, (ii) approximations when the basic process is non-Markovian, (iii) the introduction of decision variables, e.g. when to begin servicing, control of arrivals, etc. These questions, plus others, are presently under active investigation by the authors.

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Table 1: $\lambda = 0.25$ $\mu = 1$

N	5	10	15	20	25	30	35	40	45	50
E[W]	7.6	27.3	68.7	136.4	230.3	350.0	494.6	664.5	859.4	1079.4
W_D	0.1	13.7	54.0	120.6	212.7	330.1	472.5	640.0	832.5	1050.0

Table 2: $\lambda = 0.75$ $\mu = 1$

N	5	10	15	20	25	30	35	40	45	50
E[W]	10.7	43.3	101.5	184.8	293.1	426.4	584.7	768.1	976.4	1209.7
W_D	6.0	36.7	92.5	173.3	279.2	410.0	565.8	746.7	952.5	1183.3

Numerical Comparisons Between E[W] and W_D

(Deterministic Approximation)

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13. ABSTRACT

Many (perhaps most) service systems, such as repair and job shops, computation centers, and transportation networks, experience demand that is non-stationary in time. This paper describes models for situations in which demands made are by a finite number of individuals, who, having been served, do not return until much later. Such a transitory demand or arrival process describes many phenomena, among them being commuter rush hours, and also perhaps the effect on a population of individuals their simultaneous exposure to a dosage of medecine, a disease, or even a pollutant. Our paper formulates several models for the service of such demands and describes the manner in which system state may be approximated by Gaussian processes, in particular the Ornstein-Uhlenbeck and Wiener diffusions.

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